

RECIPROCITY LAWS FOR DEDEKIND SYMBOLS AND THE EULER CLASS

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ABSTRACT. We show that reciprocity laws for generalized Dedekind sums can be deduced from a concrete realization of the bounded Euler class for lattices in $\mathrm{SL}_2(\mathbb{R})$.

Asai observed that the reciprocity law for Dedekind sums follows from a certain "splitting" of the central extension

$$0 \rightarrow \mathbb{Z} \rightarrow \widetilde{\mathrm{SL}(2, \mathbb{R})} \rightarrow \mathrm{SL}(2, \mathbb{Z}) \rightarrow 1.$$

We develop and extend this point of view. First, to highlight that the splitting is provided by the explicit construction of a bounded representative of the Euler class. Then, to deduce the reciprocity of Dedekind symbols, which are natural generalizations of the Dedekind sums for lattices in $\mathrm{SL}_2(\mathbb{R})$. In doing so, we demonstrate that the reciprocity of Dedekind sums goes well beyond any arithmetical aspect of the associated group. As an application of our results, we prove combinatorial formulas for the Dedekind symbol for Hecke triangle groups.

Let $((\cdot)) : \mathbb{R} \rightarrow (-\frac{1}{2}, \frac{1}{2})$ be the sawtooth function defined by

$$((x)) = \begin{cases} x - [x] - 1/2 & x \notin \mathbb{Z}, \\ 0 & x \in \mathbb{Z}, \end{cases}$$

where $[x]$ is the largest integer $\leq x$. Dedekind [De892] introduced the arithmetic sums

$$s(d, c) = \sum_{k=1}^{c-1} \left(\left(\frac{k}{c} \right) \right) \left(\left(\frac{kd}{c} \right) \right), \quad (1)$$

where $c \in \mathbb{N}$, $(c, d) = 1$, in connection to the transformation of the logarithm of the Dedekind η -function, and deduced from that transformation the beautiful identity

$$s(d, c) + s(c, d) = \frac{1}{12} \left(\frac{d}{c} + \frac{1}{dc} + \frac{c}{d} \right) - \frac{1}{4}. \quad (2)$$

Beyond being esthetically appealing, the reciprocity law (2) has an immediate practical purpose; together with the more obvious observations

$$\begin{aligned} s(0, 1) &= 0 \quad \text{and} \\ s(d', c) &= s(d, c) \quad \text{if } d' \equiv d \pmod{c}, \end{aligned}$$

(2) allows for the fast computation of values of Dedekind sums via the Euclidean algorithm. (In terms of applications, this is all the more significant when considering, say, the various interactions between Dedekind-type sums and pseudo-random number generation.)

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Dedekind's reciprocity law (2) can be seen as a special case of the more general formula deduced by Dieter [Die57] – again from the transformation of $\log \eta$ –,

$$s(d_1, c_1) + s(d_2, c_2) + s(d_3, c_3) = \frac{1}{12} \left(\frac{c_1}{c_3 c_2} + \frac{c_2}{c_1 c_3} + \frac{c_3}{c_2 c_1} \right) - \frac{1}{4} \quad (3)$$

for c_i, d_i given by the relation

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \begin{pmatrix} a_3 & b_3 \\ c_3 & d_3 \end{pmatrix} = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix},$$

where each matrix is an element of $\mathrm{SL}_2(\mathbb{Z})$.

Following an idea of Kubota, Asai [Asa70] argues that these reciprocity formulas are consequences of a deeper mechanism underlying the relation between Dedekind sums and the transformation of $\log \eta$. More particularly, he shows that (3) can be derived without relying explicitly on the theory of modular forms, but rather by a careful investigation of the "splitting"¹ of the central extension

$$0 \rightarrow \mathbb{Z} \rightarrow \widetilde{\mathrm{SL}_2(\mathbb{R})} \rightarrow \mathrm{SL}_2(\mathbb{Z}) \rightarrow 1, \quad (4)$$

where $\widetilde{\mathrm{SL}_2(\mathbb{R})}$ denotes the universal covering group of $\mathrm{SL}_2(\mathbb{R})$.

It is a standard fact that isomorphism classes of central extensions are classified by cohomology in degree 2 (see e.g. [Bro82]), and that the second cohomology group $H^2(\mathrm{SL}_2(\mathbb{R}))$ is generated by the Euler class. In these terms, the reciprocity law (3) can be tracked down to two features of a concrete (bounded) 2-cocycle representative ω of the Euler class:

(I) the Dedekind sums are determined by a function $\rho : \mathrm{SL}_2(\mathbb{Z}) \rightarrow \mathbb{R}$ satisfying

$$\rho(\gamma\tau) - \rho(\gamma) - \rho(\tau) = \omega(\gamma, \tau); \quad (5)$$

(II) the values of ω can be easily computed.

Consequently, Asai suggests that (5) be named the generalized reciprocity law.

In this framework, we will deduce the reciprocity law (3) for Dedekind symbols attached to lattices in $\mathrm{SL}_2(\mathbb{R})$. We conclude that, while (1), (2) and (3) are arithmetic formulas, and can as well be proven purely arithmetically (see [RaG72]), their underlying mechanism is not only independent of arithmetic – that was already clear from Dedekind's original proof of (2) – but also independent of any arithmetic properties of the group at large.

Presentation of main results. If $\Gamma < \mathrm{SL}_2(\mathbb{R})$ is a lattice with cusp(s), a construction of maps $\rho : \Gamma \rightarrow \mathbb{R}$ satisfying (5) has appeared in different places in the automorphic forms literature, with connections to the Kronecker first limit formula and to the theory of multiplier systems [Gol73, Gol74, Hej83, ?]. Using this construction, the author introduced in [Bu015] the analogue of Dedekind sums for Γ . We recall its definition below.

Let $\Gamma_{\mathfrak{a}}$ be the isotropy subgroup for a cusp \mathfrak{a} for Γ . Fix a scaling $\sigma_{\mathfrak{a}}$. Each non-trivial double coset $[\gamma] = \Gamma_{\mathfrak{a}}\gamma\Gamma_{\mathfrak{a}}$, $\gamma \in \Gamma$, yields the Dedekind symbol

$$\mathcal{S}_{\mathfrak{a}}([\gamma]) = \frac{\mathrm{vol}(\Gamma \backslash \mathbb{H})}{4\pi} \frac{a_{\gamma} + d_{\gamma}}{c_{\gamma}} - \rho_{\mathfrak{a}} \begin{pmatrix} a_{\gamma} & b_{\gamma} \\ c_{\gamma} & d_{\gamma} \end{pmatrix} - \frac{1}{4} \mathrm{sign}(c_{\gamma}) \quad (6)$$

where $\begin{pmatrix} a_{\gamma} & b_{\gamma} \\ c_{\gamma} & d_{\gamma} \end{pmatrix} = \sigma_{\mathfrak{a}}^{-1} \gamma \sigma_{\mathfrak{a}}$ and where $\rho_{\mathfrak{a}}$ verifies (5). We note that the definition of Dedekind symbol does not actually depend on the particular choice of scaling $\sigma_{\mathfrak{a}}$. Alternatively, the

¹ Asai's notion of splitting differs from the usual definition; his is given by the functional equation (5) below.

double coset structure yields an equivalent definition of the Dedekind symbol as a periodic function on a restricted subset of the orbit $\mathcal{O} = \Gamma \cdot \mathbf{a}$.

Our main result expresses the reciprocity law (3) for the Dedekind symbols.

Theorem 1. *In the notation introduced above,*

$$\mathcal{S}_{\mathbf{a}}(\llbracket \gamma \rrbracket) + \mathcal{S}_{\mathbf{a}}(\llbracket \tau \rrbracket) - \mathcal{S}_{\mathbf{a}}(\llbracket \gamma\tau \rrbracket) = \frac{\text{vol}(\Gamma \backslash \mathbb{H})}{4\pi} \left(\frac{c_\gamma}{c_{\gamma\tau}c_\tau} + \frac{c_\tau}{c_\gamma c_{\gamma\tau}} + \frac{c_{\gamma\tau}}{c_\tau c_\gamma} \right) - \frac{1}{4} \text{sign}(c_\gamma c_\tau c_{\gamma\tau}), \quad (7)$$

for all $\gamma, \tau \in \Gamma$ such that $\llbracket \gamma \rrbracket, \llbracket \tau \rrbracket, \llbracket \gamma\tau \rrbracket$ are non-trivial double cosets in $\Gamma_{\mathbf{a}} \backslash \Gamma / \Gamma_{\mathbf{a}}$.

On the other hand, the generalization of Dedekind's reciprocity law (2) only makes sense for groups that contain the involution $S = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$. In the second part of this article, we restrict our attention to the Hecke triangle groups G_q . We recall that G_q is a discrete triangle group of type $(2, q, \infty)$ for $q \geq 3$. This means that G_q is generated by reflections on the sides of a triangle with interior angles $(\pi/2, \pi/q, 0)$. Algebraically, G_q is generated by the involution S together with the translation $T_q = \begin{pmatrix} 1 & \lambda_q \\ & 1 \end{pmatrix}$, where $\lambda_q = 2 \cos(\pi/q)$. Since there is only one cusp, hence one Dedekind symbol, we will write \mathcal{S} instead of \mathcal{S}_∞ .

Theorem 2. *For each $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_q$, the following reciprocity law holds*

$$\mathcal{S}(\llbracket \gamma \rrbracket) - \mathcal{S}(\llbracket \gamma S \rrbracket) = \frac{1 - 2/q}{8 \cos(\pi/q)} \left(\frac{d}{c} + \frac{1}{dc} + \frac{c}{d} \right) - \frac{1}{4} \text{sign}(cd) \quad (8)$$

whenever $\llbracket \gamma \rrbracket, \llbracket \gamma S \rrbracket$ are non-trivial.

For the Hecke triangle group G_q , the Dedekind symbol \mathcal{S} can equivalently be seen as a function on $\mathcal{O}' = \{\gamma \cdot \infty \bmod \lambda_q : \gamma \in G_q\}$. In this setting, \mathcal{S} can be expressed in terms of λ_q -continued fractions. In fact, each element $\gamma \in G_q$ can be expressed as a word in the group generators $S_q = \begin{pmatrix} 1 & \lambda_q \\ \lambda_q & 1 \end{pmatrix}$ and $T_q = \begin{pmatrix} 1 & \lambda_q \\ & 1 \end{pmatrix}$, and

$$(S_q^{a_1} T_q^{a_2} \dots T_q^{a_{n-1}} S_q^{a_n}) \cdot \infty = \frac{1}{a_1 \lambda_q + \frac{1}{a_2 \lambda_q + \frac{1}{\dots}}} =: [0, a_1, a_2, \dots, a_n]$$

By applying the reciprocity law recursively, we obtain an explicit formula for the Dedekind symbol \mathcal{S} in terms of λ_q -continued fractions, which generalizes a theorem of Hickerson for the Dedekind sums [Hic77, Thm. 1].

Theorem 3. *For each $\gamma \in G_q$,*

$$\llbracket \gamma \rrbracket = \begin{cases} \llbracket S_q^{a_1} T_q^{a_2} \dots T_q^{a_{n-1}} S_q^{a_n} \rrbracket & \text{if } n \text{ is odd,} \\ \llbracket S_q^{a_1} T_q^{a_2} \dots T_q^{a_n} S \rrbracket & \text{if } n \text{ is even,} \end{cases}$$

for a uniquely determined finite sequence $(a_n) \subset \mathbb{N}$. Then

$$\mathcal{S}(\llbracket \gamma \rrbracket) = \frac{1 - 2/q}{8 \cos(\pi/q)} \left([0 : a_1, \dots, a_n] + (-1)^{n+1} [0 : a_n, \dots, a_1] - \sum_{j=1}^n (-1)^j a_j \lambda_q \right) - \frac{1 - (-1)^n}{8}.$$

0. NOTATION AND TERMINOLOGY

We let \mathbb{H} denote the hyperbolic upper half-plane, and recall that $\mathrm{SL}_2(\mathbb{R})$ acts on \mathbb{H} by fractional linear transformations, and that this action factors through $\mathrm{PSL}_2(\mathbb{R})$. Throughout the article, Γ will denote a cofinite Fuchsian group. That is, a discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$ of finite covolume $V = \mathrm{vol}(\Gamma \backslash \mathbb{H}) < \infty$, containing at least one parabolic element. We recall that $\gamma \in \Gamma$ is parabolic if $|\mathrm{tr}(\gamma)| = 2$ or, equivalently, if the action of γ on $\overline{\mathbb{H}} = \mathbb{H} \cup \partial\mathbb{H}$ fixes a single point and that this point is in $\partial\mathbb{H} = \mathbb{R} \cup \{\infty\}$. Such a point is referred to as a cusp and will be denoted \mathfrak{a} . For each cusp \mathfrak{a} , the isotropy subgroup of elements of Γ fixing \mathfrak{a} is denoted by $\Gamma_{\mathfrak{a}}$. It is, up to sign, an infinite cyclic subgroup of Γ . A matrix $\sigma_{\mathfrak{a}} \in \mathrm{SL}(2, \mathbb{R})$ is called a scaling, if it verifies $\sigma_{\mathfrak{a}}(\infty) = \mathfrak{a}$ and

$$\sigma_{\mathfrak{a}}^{-1} \Gamma_{\mathfrak{a}} \sigma_{\mathfrak{a}} = (\sigma_{\mathfrak{a}}^{-1} \Gamma_{\mathfrak{a}})_{\infty} = \pm \begin{pmatrix} 1 & \mathbb{Z} \\ & 1 \end{pmatrix}.$$

These two conditions do not determine uniquely $\sigma_{\mathfrak{a}}$ but up to right multiplication by any element of the group $\begin{pmatrix} 1 & \mathbb{R} \\ & 1 \end{pmatrix}$. There is a one-to-one correspondence between subgroups of $\mathrm{PSL}_2(\mathbb{R})$ and subgroups of $\mathrm{SL}_2(\mathbb{R})$ that contain $-I = \begin{pmatrix} -1 & \\ & -1 \end{pmatrix}$. We shall always assume that $-I \in \Gamma$.

1. A CONCRETE REALIZATION OF THE BOUNDED EULER CLASS

Petersson's cocycle. We review the classical construction investigated by Petersson [Pet30, Pet38], which provides the explicit bounded Euler class representative ω . For any $z \in \mathbb{H}$, set

$$\omega(g, h) = \frac{1}{2\pi i} (\log j(g, hz) + \log j(h, z) - \log j(gh, z)), \quad (9)$$

where \log denotes the principal branch of the logarithm, that is $\log(cz + d) = \ln|cz + d| + i \arg(cz + d)$ for $-\pi < \arg(cz + d) \leq \pi$, and where $j(g, z) = cz + d$, $g = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$, is the usual automorphy factor, which satisfies

$$j(gh, z) = j(g, hz)j(h, z). \quad (10)$$

Lemma 1.1. *The function ω defines a bounded 2-cocycle representative of the Euler class.*

Proof. Observe that (10) implies that the LHS of (9) is real-valued. Since it is also holomorphic in z , it must be constant; ω is indeed independent of the choice of z . By definition, ω can only take integer values. Since moreover $|\omega(g, h)| \leq \frac{3}{2}$, we conclude that ω only takes values in $\{-1, 0, 1\}$. Finally, one can check by direct computation that

$$\omega(g_1, g_2) + \omega(g_1 g_2, g_3) = \omega(g_1, g_2 g_3) + \omega(g_2, g_3).$$

Hence $[\omega] \in H^2(\mathrm{SL}_2(\mathbb{R}))$ and we can conclude from the previous observations that $[\omega]$ coincides with the Euler class. \square

Computing values of ω . The 2-cocycle (9) can be computed on explicit elements. Asai provides a clean formula to do so [Asa70, Thm. 2], simplifying the laborious presentation of Petersson [Pet38]. However, his formula is obtained under the unusual choice $-\pi \leq \arg(cz + d) < \pi$. By a simple reparametrization, we recover that formula for the principal branch of the logarithm.

Theorem 1.2. *Set*

$$c(-d) = \begin{cases} c & c \neq 0, \\ -d & c = 0, \end{cases} \quad \text{and} \quad \text{sign}(x) = \begin{cases} 1 & x > 0, \\ 0 & x = 0, \\ -1 & x < 0. \end{cases}$$

The explicit values of $\omega(g, h)$ are given by the following table.

$\text{sign}(c_g(-d_g))$	$\text{sign}(c_h(-d_h))$	$\text{sign}(c_{gh}(-d_{gh}))$	$\omega(g, h)$
1	1	-1	1
-1	-1	1	-1
<i>otherwise</i>			0

(11)

Proof. One can check directly the validity of the identity

$$\log(cz + d) = \log\left(\frac{cz + d}{i \text{sign}(c(-d))}\right) + i\frac{\pi}{2}\text{sign}(c(-d)).$$

Showing by inspection that

$$\begin{aligned} & \log\left(\frac{j(g, hz)}{i \text{sign}(c_g(-d_g))}\right) + \log\left(\frac{j(h, z)}{i \text{sign}(c_h(-d_h))}\right) - \log\left(\frac{j(gh, z)}{i \text{sign}(c_{gh}(-d_{gh}))}\right) \\ &= -i\frac{\pi}{2}\text{sign}(c_g(-d_g)c_h(-d_h)c_{gh}(-d_{gh})), \end{aligned}$$

we obtain the formula

$$\omega(g, h) = \frac{1}{4} \{ \text{sign}(c_g(-d_g)) + \text{sign}(c_h(-d_h)) - \text{sign}(c_{gh}(-d_{gh})) - \text{sign}(c_g(-d_g)c_h(-d_h)c_{gh}(-d_{gh})) \}$$

from which it is easy to complete table (11). \square

2. RECIPROCITY OF DEDEKIND SYMBOLS

Definition of Dedekind symbols. Let Γ be a cofinite Fuchsian group with a cusp \mathfrak{a} and fix a scaling $\sigma_{\mathfrak{a}}$. In [Bu015], we introduced the Dedekind symbol $\mathcal{S}_{\mathfrak{a}}$, which we recall is defined on non-trivial double cosets of $\Gamma_{\mathfrak{a}} \backslash \Gamma / \Gamma_{\mathfrak{a}}$ by

$$\mathcal{S}_{\mathfrak{a}}([\gamma]) = \frac{\text{vol}(\Gamma \backslash \mathbb{H})}{4\pi} \frac{a_{\gamma} + d_{\gamma}}{c_{\gamma}} - \rho_{\mathfrak{a}} \begin{pmatrix} a_{\gamma} & b_{\gamma} \\ c_{\gamma} & d_{\gamma} \end{pmatrix} - \frac{1}{4}\text{sign}(c_{\gamma})$$

for $\begin{pmatrix} a_{\gamma} & b_{\gamma} \\ c_{\gamma} & d_{\gamma} \end{pmatrix} = \sigma_{\mathfrak{a}}^{-1} \gamma \sigma_{\mathfrak{a}}$, and $\rho_{\mathfrak{a}} : \sigma_{\mathfrak{a}}^{-1} \Gamma \sigma_{\mathfrak{a}} \rightarrow \mathbb{R}$ satisfying

$$\rho_{\mathfrak{a}}(\gamma\tau) - \rho_{\mathfrak{a}}(\gamma) - \rho_{\mathfrak{a}}(\tau) = \omega(\gamma, \tau), \quad (12)$$

where ω is the bounded 2-cocycle given by (9), and the definition does not depend on the particular scaling $\sigma_{\mathfrak{a}}$ [Bu015, Thm. 2]. We will not review here the construction of $\rho_{\mathfrak{a}}$ or the relation of Dedekind symbols to automorphic forms²; we refer the reader to [Bu015].

The most important characteristic of this definition is that the Dedekind symbols factor through (non-trivial) double cosets in $\Gamma_{\mathfrak{a}} \backslash \Gamma / \Gamma_{\mathfrak{a}}$. By conjugation with $\sigma_{\mathfrak{a}}$, a double coset representative γ of $[\gamma]$ gets sent to

$$(\sigma_{\mathfrak{a}}^{-1} \Gamma_{\mathfrak{a}} \sigma_{\mathfrak{a}}) \begin{pmatrix} a_{\gamma} & b_{\gamma} \\ c_{\gamma} & d_{\gamma} \end{pmatrix} (\sigma_{\mathfrak{a}}^{-1} \Gamma_{\mathfrak{a}} \sigma_{\mathfrak{a}}) = \pm \begin{pmatrix} a_{\gamma} + c_{\gamma} \mathbb{Z} & * \\ c_{\gamma} & d_{\gamma} + c_{\gamma} \mathbb{Z} \end{pmatrix}. \quad (13)$$

In consequence, we observe that $[\gamma]$ is non-trivial if and only if $c_{\gamma} \neq 0$. Moreover, one can always choose a representative γ of $[\gamma]$ such that $c_{\gamma} > 0$, and $0 \leq a_{\gamma}, d_{\gamma} < c_{\gamma}$. These

² In the form of generalized log η -functions.

two simple observations are primordial in establishing the Dedekind symbol as the natural generalization of the Dedekind sums. (On $\mathrm{SL}_2(\mathbb{Z})$, the two definitions coincide.)

Alternatively (and equivalently), the Dedekind symbol $\mathcal{S}_{\mathfrak{a}}$ can be defined as a periodic function on the orbit (or cusp set) $\mathcal{O}' = \Gamma \cdot \mathfrak{a} \setminus \{\mathfrak{a}\}$. Define an equivalence relation on \mathcal{O} that identifies $x, y \in \mathcal{O}$ if there exists some $\gamma \in \Gamma_{\mathfrak{a}}$ such that $\gamma x = y$.

Theorem 2.1. *For any $\gamma \in \Gamma$ such that $[\![\gamma]\!]$ is non-trivial, $\mathcal{S}_{\mathfrak{a}}([\gamma \cdot \mathfrak{a}]) = \mathcal{S}_{\mathfrak{a}}([\![\gamma]\!])$.*

Proof. Consider the double coset on the LHS of (13). Let $\begin{pmatrix} a_{\gamma} & b \\ c_{\gamma} & d \end{pmatrix}, \begin{pmatrix} a' & b' \\ c_{\gamma} & d_{\gamma} \end{pmatrix}$ be two other representative of $[\![\gamma]\!]$. Then

$$\begin{pmatrix} a_{\gamma} & b \\ c_{\gamma} & d \end{pmatrix}^{-1} \begin{pmatrix} a_{\gamma} & b_{\gamma} \\ c_{\gamma} & d_{\gamma} \end{pmatrix}, \begin{pmatrix} a_{\gamma} & b_{\gamma} \\ c_{\gamma} & d_{\gamma} \end{pmatrix} \begin{pmatrix} a' & b' \\ c_{\gamma} & d_{\gamma} \end{pmatrix}^{-1} = \begin{pmatrix} 1 & * \\ & 1 \end{pmatrix} \in (\sigma_{\mathfrak{a}}^{-1} \Gamma_{\mathfrak{a}} \sigma_{\mathfrak{a}}).$$

That is, any double coset representative of $[\![\gamma]\!]$ is completely determined by either its first column or its first row. Therefore, the double cosets $[\![\begin{pmatrix} a_{\gamma} & * \\ c_{\gamma} & * \end{pmatrix}]\!] \in (\sigma_{\mathfrak{a}}^{-1} \Gamma_{\mathfrak{a}} \sigma_{\mathfrak{a}}) \setminus (\sigma_{\mathfrak{a}}^{-1} \Gamma \sigma_{\mathfrak{a}}) / (\sigma_{\mathfrak{a}}^{-1} \Gamma_{\mathfrak{a}} \sigma_{\mathfrak{a}})$ are in one-to-one correspondence with the cusp points $[a_{\gamma}/c_{\gamma}] = a_{\gamma}/c_{\gamma} \pmod{1} \in (\sigma_{\mathfrak{a}}^{-1} \Gamma_{\mathfrak{a}} \sigma_{\mathfrak{a}}) \setminus (\sigma_{\mathfrak{a}}^{-1} \Gamma \sigma_{\mathfrak{a}}) \cdot \infty$. \square

Selected properties.

Lemma 2.2. *For any non-trivial double coset $[\![\gamma]\!]$, $\mathcal{S}_{\mathfrak{a}}(-[\![\gamma]\!]) = \mathcal{S}_{\mathfrak{a}}([\![\gamma]\!])$.*

Proof. Following the discussion above, we may assume that $c > 0$. Using (12),

$$\rho_{\mathfrak{a}}\left(-\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) - \frac{1}{4} \mathrm{sign}(-c) = \rho_{\mathfrak{a}}(-I) + \rho_{\mathfrak{a}}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) + \omega\left(-I, \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) + \frac{1}{4}.$$

With (11), we can check that

$$\begin{aligned} \rho_{\mathfrak{a}}(I) &= -\omega(I, I) = 0, \\ 2\rho_{\mathfrak{a}}(-I) &= \rho_{\mathfrak{a}}(I) - \omega(-I, -I) = -1, \\ \text{and } \omega(-I, \begin{pmatrix} a & b \\ c & d \end{pmatrix}) &= 0, \end{aligned}$$

The statement follows. \square

Lemma 2.3. *For any non-trivial double coset $[\![\gamma]\!]$, $\mathcal{S}_{\mathfrak{a}}([\![\gamma]\!]) = -\mathcal{S}_{\mathfrak{a}}([\![\gamma^{-1}]\!])$.*

Proof. It is enough to show that $\rho_{\mathfrak{a}}\left(\begin{pmatrix} a_{\gamma} & b_{\gamma} \\ c_{\gamma} & d_{\gamma} \end{pmatrix}\right)^{-1} = -\rho_{\mathfrak{a}}\left(\begin{pmatrix} a_{\gamma} & b_{\gamma} \\ c_{\gamma} & d_{\gamma} \end{pmatrix}\right)$. By (12), $\rho_{\mathfrak{a}}(g) + \rho_{\mathfrak{a}}(g^{-1}) = \omega(g, g^{-1})$ and (11) yields $\omega(g, g^{-1}) = 0$. \square

With this last identity, we can see that the reciprocity law (7) expressed in Theorem 1 corresponds exactly to (3). In exactly the same fashion, we deduce (7) from (12).

Generalized reciprocity law for Dedekind symbols.

Proof of Theorem 1. For $c_{\gamma}c_{\tau}c_{\gamma\tau} \neq 0$,

$$\begin{aligned} \frac{a_{\gamma} + d_{\gamma}}{c_{\gamma}} + \frac{a_{\tau} + d_{\tau}}{c_{\tau}} - \frac{a_{\gamma\tau} + d_{\gamma\tau}}{c_{\gamma\tau}} &= \frac{a_{\gamma} + d_{\gamma}}{c_{\gamma}} + \frac{a_{\tau} + d_{\tau}}{c_{\tau}} - \frac{(a_{\gamma}a_{\tau} + b_{\gamma}c_{\tau}) + (c_{\gamma}b_{\tau} + d_{\gamma}d_{\tau})}{c_{\gamma}a_{\tau} + d_{\gamma}c_{\tau}} \\ &= \frac{c_{\gamma}^2 + c_{\tau}^2 + (c_{\gamma}a_{\tau} + d_{\gamma}c_{\tau})^2}{c_{\gamma}c_{\tau}(c_{\gamma}a_{\tau} + d_{\gamma}c_{\tau})} = \frac{c_{\gamma}^2 + c_{\tau}^2 + c_{\gamma\tau}^2}{c_{\gamma}c_{\tau}c_{\gamma\tau}} \end{aligned}$$

and thus, by (12),

$$\begin{aligned} \mathcal{S}_a(\llbracket \gamma \rrbracket) + \mathcal{S}_a(\llbracket \tau \rrbracket) - \mathcal{S}_a(\llbracket \gamma\tau \rrbracket) &= \frac{V}{4\pi} \left(\frac{c_\gamma}{c_{\gamma\tau}c_\tau} + \frac{c_\tau}{c_\gamma c_{\gamma\tau}} + \frac{c_{\gamma\tau}}{c_\tau c_\gamma} \right) + \omega(\gamma, \tau) + \\ &\quad + \frac{1}{4} (\text{sign}(c_{\gamma\tau}) - \text{sign}(c_\gamma) - \text{sign}(c_\tau)) \\ &= \frac{V}{4\pi} \left(\frac{c_\gamma}{c_{\gamma\tau}c_\tau} + \frac{c_\tau}{c_\gamma c_{\gamma\tau}} + \frac{c_{\gamma\tau}}{c_\tau c_\gamma} \right) - \frac{1}{4} \text{sign}(c_\gamma c_\tau c_{\gamma\tau}) \end{aligned}$$

as in the proof of Theorem 1.2. \square

3. EXPLICIT FORMULAS FOR HECKE TRIANGLE GROUPS

Parametrization. Fix the scaling $\sigma_\infty = \begin{pmatrix} \sqrt{\lambda_q} & \\ & 1/\sqrt{\lambda_q} \end{pmatrix}$ and set

$$\begin{aligned} s &:= \sigma_\infty^{-1} S \sigma_\infty = \begin{pmatrix} & -1/\lambda_q \\ \lambda_q & \end{pmatrix}, \\ u &:= \sigma_\infty^{-1} T_q^n \sigma_\infty = \begin{pmatrix} 1 & n \\ & 1 \end{pmatrix}, \\ v &:= \sigma_\infty^{-1} S_q^n \sigma_\infty = \begin{pmatrix} 1 & \\ n\lambda_q^2 & 1 \end{pmatrix}, \\ \gamma_q &:= \sigma_\infty^{-1} \gamma \sigma_\infty = \begin{pmatrix} a & b/\lambda_q \\ c\lambda_q & d \end{pmatrix}, \end{aligned}$$

for any $n \in \mathbb{Z}$ and any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_q$. We record the following formulas for later computations. Let $V_q = \text{vol}(G_q \backslash \mathbb{H})$. Using the Gauss–Bonnet formula, $V_q = \pi(1 - 2/q)$.

Lemma 3.1. *For each $n \in \mathbb{Z}$,*

$$\rho(u^n) = \frac{V_q n}{4\pi}.$$

Proof. This is Lemma 4.2 in [Bu015]. \square

Lemma 3.2. *Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{R})$. Then*

$$\begin{aligned} \omega(g, s) &= \begin{cases} 1 & \text{if } c > 0 \text{ and } d < 0, \\ 0 & \text{otherwise;} \end{cases} \\ \omega(g, u^n) &= 0 \text{ for all } n \in \mathbb{Z} \\ \rho(s) &= -1/4 \\ \rho(v^n) &= -\rho(u^n) \text{ for all } n \in \mathbb{Z}. \end{aligned}$$

Proof. Follows from Theorem 1.2. \square

Dedekind's reciprocity law for Hecke triangle groups.

Proof of Theorem 2. Choose a double coset representative γ such that $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has matrix entries $c > 0$ and $0 < d < c$. Note that

$$\gamma_q s = \begin{pmatrix} b & -a/\lambda_q \\ d\lambda_q & -c \end{pmatrix}.$$

Proceeding as in the proof of Theorem 1, we obtain

$$\begin{aligned}\mathcal{S}(\llbracket \gamma \rrbracket) - \mathcal{S}(\llbracket \gamma S \rrbracket) &= \frac{V_q}{4\pi\lambda_q} \left(\frac{d}{c} + \frac{1}{dc} + \frac{c}{d} \right) + \rho(s) + \omega(\gamma_q, s) - \frac{1}{4} (\text{sign}(c) - \text{sign}(d)) \\ &= \frac{1 - 2/q}{8 \cos(\pi/q)} \left(\frac{d}{c} + \frac{1}{dc} + \frac{c}{d} \right) - \frac{1}{4} \text{sign}(cd).\end{aligned}$$

□

A formula for the Dedekind symbol attached to a Hecke triangle group.

Proof of Theorem 3. We proceed by induction. If $n = 1$, then, by definition,

$$\mathcal{S}(\llbracket S_q^{a_1} \rrbracket) = \frac{V_q}{4\pi} \frac{a+d}{c\lambda_q} - \rho(v^{a_1}) - \frac{1}{4} = \frac{V_q}{4\pi} \frac{a+d}{c\lambda_q} + a_1 - \frac{1}{4},$$

and, we can check with Lemma 3.2 that

$$\mathcal{S}(\llbracket S_q^{a_1} T_q^{a_2} S \rrbracket) = \frac{V_q}{4\pi} \frac{a+d}{c\lambda_q} - \rho(v^{a_1} u^{a_2} s) - \frac{1}{4} = \frac{V_q}{4\pi} \frac{a+d}{c\lambda_q} + a_1 - a_2$$

for $n = 2$. For $n \geq 3$, let

$$\gamma_n = \begin{cases} S_q^{a_1} T_q^{a_2} \dots T_q^{a_{n-1}} S_q^{a_n} & \text{if } n \text{ is odd,} \\ S_q^{a_1} T_q^{a_2} \dots T_q^{a_n} S & \text{if } n \text{ is even,} \end{cases}$$

and observe that

$$\llbracket \gamma_{n-1} \rrbracket = \begin{cases} \llbracket \gamma_n S T_q^{a_n} \rrbracket = \llbracket \gamma_n S \rrbracket & \text{if } n \text{ is odd,} \\ \llbracket \gamma_n S T_q^{-a_n} \rrbracket = \llbracket \gamma_n S \rrbracket & \text{if } n \text{ is even.} \end{cases}$$

By our induction hypothesis,

$$\mathcal{S}(\llbracket \gamma_{n-1} \rrbracket) = \frac{V_q}{4\pi\lambda_q} \left(\gamma_{n-1}.\infty - \gamma_{n-1}^{-1}.\infty - \sum_{j=1}^{n-1} (-1)^j a_j \lambda_q \right) - \frac{1 - (-1)^{n-1}}{8},$$

where

$$\gamma_{n-1}.\infty - \gamma_{n-1}^{-1}.\infty = (\gamma_n S).\infty - (T_q^{\mp a_n} S \gamma_n^{-1}).\infty = (\gamma_n S).\infty \pm a_n \lambda_q - (S \gamma_n^{-1}).\infty$$

with $+a_n \lambda_q$ if n is odd and $-a_n$ otherwise. Applying the reciprocity law,

$$\begin{aligned}\mathcal{S}(\llbracket \gamma_n \rrbracket) &= \frac{V_q}{4\pi\lambda_q} \left(-\gamma_n^{-1}.\infty + \frac{1}{cd} + S \gamma_n^{-1}.\infty \right) - \frac{1}{4} \text{sign}(cd) + \mathcal{S}(\llbracket \gamma_{n-1} \rrbracket) \\ &= \frac{V_q}{4\pi\lambda_q} \left(\frac{d}{c} + \frac{1}{cd} + \frac{b}{d} - \sum_{j=1}^n (-1)^j a_j \lambda_q \right) - \frac{1 - (-1)^{n-1}}{8} \\ &= \frac{V_q}{4\pi\lambda_q} \left(\frac{a+d}{c} - \sum_{j=1}^n (-1)^j a_j \lambda_q \right) - \frac{1 - (-1)^{n-1}}{8}\end{aligned}$$

and the identification with continued fraction expansions is immediate considering $\frac{a}{c} = \gamma_n.\infty$ and $\frac{d}{c} = -\gamma_n^{-1}.\infty$.

□

REFERENCES

- [Asa70] T. Asai, *The reciprocity of Dedekind sums and the factor set for the universal covering group of $SL(2, \mathbb{R})$* . Nagoya Math. J. **37** (1970), 67–80.
- [Bro82] K. S. Brown, *Cohomology of groups*. Grad. Texts in Math. Vol. 87, Springer-Verlag, New York, 1982.
- [Bu015] C. Burrin, *Generalized Dedekind sums and equidistribution mod 1*. [arXiv:1509.04429](https://arxiv.org/abs/1509.04429)
- [De892] R. Dedekind, *Erläuterungen zu den Fragmenten XXVIII*. In B. Riemann, Ges. Math. Werke, Leipzig, 1892, 466–478; Dedekind Ges. Werke I, 1930, 159–172.
- [Die57] U. Dieter, *Beziehungen zwischen Dedekindschen Summen*. Abh. Math. Sem. Hamburg **21** (1957), 109–125.
- [Gol73] L.J. Goldstein, *Dedekind sums for a Fuchsian group, I*. Nagoya Math. J. **50** (1973), 21–47.
- [Gol74] L.J. Goldstein, *Errata for "Dedekind sums for a Fuchsian group, I."* Nagoya Math. J. **53** (1974), 235–237.
- [Hej83] D. Hejhal, *The Selberg trace formula for $PSL(2, \mathbb{R})$, Vol. 2*. Lect. Notes in Math. **1001**, Springer-Verlag, 1983.
- [Hic77] D. Hickerson, *Continued fractions and density results for Dedekind sums*. J. f. d. reine u. angew. Math. (1977), 113–116.
- [Pet30] H. Petersson, *Theorie der automorphen Formen beliebiger reeller Dimension und ihre Darstellung durch eine neue Art Poincaréscher Reihen*. Math. Ann. **103** (1930), 396–436.
- [Pet38] H. Petersson, *Zur analytischen Theorie der Grenzkreisgruppen I*. Math. Ann. **115** (1938), 23–67.
- [RaG72] H. Rademacher, E. Grosswald, *Dedekind sums*. Carus Math. Monog., MAA, 1972.

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